

ON THE MAXIMAL CONTAINMENTS OF LOWER CENTRAL SERIES IDEALS

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ABSTRACT. We begin a discussion about the maximal containments of lower central series ideals: ideals generated by products of two-sided ideals of the lower central series of the free associative algebra on n generators. We introduce two new ideas to the topic, the PBW grading and the pigeonhole principle, that help us give a complete classification of the containments for $n = 2$ and obtain partial results in the general case.

1. INTRODUCTION

1.1. Preliminaries. In this paper all vector spaces are taken over a field \mathcal{K} of characteristic zero. Let A_n (simply A if n is not specified) be the free associative unitary algebra on n generators x_1, \dots, x_n . An *element* means a polynomial in A_n if not specified. The bracket operation $[a, b] = ab - ba$ equips A_n with the structure of a Lie algebra. We norm commutators to the right, i.e.

$$[a_1, a_2, \dots, a_m] \equiv [a_1, [a_2, \dots, a_m]].$$

A *pure commutator* is a commutator with single generators in its slots. For example, $[x_1, x_2, x_3]$ is a pure commutator, while $[x_1 + 7x_2, 3x_1^2x_3, x_4, 2x_5x_6]$ is not. The *length* of a commutator is its number of slots. In the above example the latter commutator has length four. We also set the following convention: a generator x_i in A_n is itself a pure commutator of length one. A *pure-commutator product* is a finite product of pure commutators. For example, $x_1x_2x_3^2[x_1, x_5, x_7][x_1, x_2][x_1, x_7]$ is a pure-commutator product consisting of 7 commutators (x_1, x_2 , two pure commutators $x_3, [x_1, x_5, x_7], [x_1, x_2]$ and $[x_1, x_7]$), while $x_1[x_1, x_2] + x_6[x_2, x_3][x_3, x_4]$ and $x_1[x_1, x_2][x_1, x_2][x_1, x_2^2]$ are not such products. We do not count 0 as a pure commutator, nor a pure-commutator product, i.e. the commutators in the product cannot consist only of one generator, unless they are of length one.

Unless it is specified we do not care about the specific generators in the commutators in pure products, and often we will only introduce a sequence (i_1, \dots, i_k) , where k is the number of commutators in the product and i_j is the length of a pure commutator for $j = 1, \dots, k$. The above-mentioned example of pure-commutator product corresponds to the 7-tuple $(1, 1, 1, 1, 3, 2, 2)$.

1.2. Motivation. The *lower central series* $\{L_i(A_n)\}_{i \geq 1}$ of A_n is defined by

$$L_1(A_n) := A_n \text{ and } L_i(A_n) := [A, L_{i-1}(A_n)]$$

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for $i \geq 2$. Unless n is specified $L_i \equiv L_i(A_n)$, and essentially L_i is spanned by all commutators of length i with entries in A_n . The series $\{M_i(A_n)\}_{i \geq 1}$ is the series of the two-sided (one sided due to the Leibniz rule) ideals $M_i = A \cdot L_i \cdot A$ generated by L_i . Note that we can abbreviate $M_i \equiv M_i(A_n)$.

The lower central series ideals form an important filtration \mathcal{M} in non-commutative algebra

$$A = M_1 \supset M_2 \supset M_3 \cdots$$

It helps us “approximate” the algebra A through the quotients $N_i := M_i/M_{i+1}$, i.e.

$$A = \text{Gr}_{\mathcal{M}}(A) = \bigoplus_{i \geq 1}^{\infty} M_i/M_{i+1} = \bigoplus_{i \geq 1}^{\infty} N_i$$

Therefore, in order to better understand this commutator-graded structure of A , we need to study the N -ideals, i.e. to be aware of the structure of the M -ideals. This gives rise to the consideration of containment of products of M -elements. By *lower central series ideals* in this paper we mean finite products of M -elements $M_{i_1} \cdots M_{i_k}$, where i_p is greater or equal to two for $p = 1, \dots, k$. There are a few results on inclusions of products of M -elements:

Theorem 1.1 (Jennings, 1947, [9]). *For every n and k there exists m , such that*

$$(M_2(A_n))^m \subset M_k(A_n).$$

Theorem 1.2 (Latyshev, 1965, [10], and Gupta and Levin, 1983, [8]). *For all m and l*

$$M_m \cdot M_l \subset M_{m+l-2}.$$

Theorem 1.3 (Bapat and Jordan, 2010, [1]). *For odd m or odd l*

$$M_m \cdot M_l \subset M_{m+l-1}.$$

Theorem 1.4 (Bapat and Jordan, 2010, [1]). *For odd l*

$$[M_l, L_k] \subset L_{k+l}.$$

Remark 1.5. In [2] (Section 1. Introduction) one may find a brief historical discussion of these results. In particular, we thank the authors for pointing out that it was Latyshev to first prove Theorem 1.2.

Essentially, these results allow us not to be redundant, i.e. to omit a few inductions that can be found in the paper of Bapat and Jordan [1]. None of the known results ask whether the inclusions are maximal. This gives rise to the discussion that follows.

We say that a pure-commutator product p has a *maximal inclusion* in $M_l(A_n)$ if p is in $M_l(A_n)$ but p is not in $M_{l+1}(A_n)$. What follows is the main definition of the paper.

Definition 1.6. For an associative unitary algebra R and a k -tuple $i = (i_1, \dots, i_k)$, such that $i_p \geq 2$ for $p = 1, \dots, k$ let

$$I(R, i) \in \mathbb{N}$$

be the maximal index such that

$$M_{i_1}(R) \cdots M_{i_k}(R) \subset M_{I(R, i)}(R),$$

while the product $M_{i_1}(R) \cdots M_{i_k}(R)$ is not contained in $M_{I(R, i)+1}(R)$.

For example, later in this paper we show that $I(A_2, (2, 2)) = 3$, i.e. $M_2(A_2) \cdot M_2(A_2) \subset M_3(A_2)$, but $M_2(A_2) \cdot M_2(A_2) \not\subset M_4(A_2)$. Note that in this example $R = A_2$ and $i = (2, 2)$ is a 2-tuple. The only results from the above-mentioned that we can apply is Theorem 1.2, which only tells us the trivial

$$M_2(A_2) \cdot M_2(A_2) \subset M_2(A_2).$$

1.3. Main results. The results presented in the Motivation imply the following bounds:

- Theorem 1.2 implies $I(A_n, (m, l)) \geq m + l - 2$;
- Theorem 1.3 implies $I(A_n, (m, l)) \geq m + l - 1$ when m or l is odd.

To our knowledge, prior to this paper there are no attempts to obtain an upper bound on $I(A_n, i)$. In this paper we find an upper bound, which tells us that $I(A_n, (m, l)) \geq m + l - 1$ is sharp for some partial cases. More specifically, we provide a complete classification of the lower central series containments for A_2 .

Theorem 1.7 (Complete classification for $n = 2$). *Given the k -tuple $i = (i_1, \dots, i_k)$, such that i_p is greater than or equal to two for $p = 1, \dots, k$ the following holds*

$$I(A_2, i) = i_1 + \dots + i_k - k + 1.$$

This is our main theorem. Finding $I(R, i)$ would characterize the *infrastructure* of the lower central series ideals in R in more detail. First of all, in Section 2 through the PBW theorem we describe an unconventional grading of A_n that helps us prove non-inclusions. Second of all, in Section 3 we obtain Theorem 1.7 via the simple pigeonhole principle. Moreover, we prove that the ideals $M_r(A_2)$ are finitely generated as two-sided ideals. We give an explicit set of generators. We conclude the section with a result on universal Lie nilpotent associative algebras. In Section 4 we discuss the case of $n = 3$. In Section 5 we present an idea of non-containment, as well as a conjecture for maximal inclusions for the case of the tuple $(2, \dots, 2) \in \mathbb{N}^k$. In Section 6 we combine the theorems, known previously, and our results. In Section 7 briefly we show applications to PI theory of the results in this paper.

2. THE PBW GRADING. NON-CONTAINMENT OF IDEALS.

Our proofs of non-containment rely on a nontrivial PBW grading of A_n .

Let \mathfrak{g}_n be the free unitary Lie algebra on n generators x_1, \dots, x_n with an ordering \leq of the basis $B = \{1, x_1, \dots, x_n, [x_1, x_2], [x_1, [x_1, x_2]], \dots\}$, in which the pure commutators correspond to Lyndon words. Note that the bracket $[\cdot, \cdot]$ of \mathfrak{g}_n is not necessarily defined by $[a, b] = ab - ba$. According to the PBW Theorem, the set $\{b_{i_1} \cdots b_{i_m} \mid m \geq 0, b_{i_j} \in B, j \leq m, i_1 \leq \dots \leq i_m\}$ forms a basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_n)$. The PBW filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots$$

is defined by $\mathcal{F}_0 = \mathcal{K}$, and $\mathcal{F}_r = \{a_1 \cdots a_k \mid a_i \in \mathfrak{g}_n, k \leq r\}$ for r greater than or equal to one. Hence, from the fact that both of the algebras (as vector spaces) satisfy the same universal property $\text{Gr}_{\mathcal{F}}(\mathcal{U}(\mathfrak{g}_n)) \cong \mathcal{K}[x_1, \dots, x_n]$. There is a natural isomorphism between A_n and $\mathcal{U}(\mathfrak{g}_n)$ defined by mapping generators. The isomorphism equips A_n with PBW grading as follows: \mathcal{K} sits in degree zero; \mathfrak{g}_n sits in PBW degree one; a product of r elements in \mathfrak{g}_n sits in PBW degree r for r greater than or equal to two.

For example, for the ordering \leq the monomial basis of A_2 up to (normal, monomial) degree three is

Monomial degree	Elements
Zero	1
One	x_1, x_2
Two	$x_1^2, x_2^2, x_1x_2, x_2x_1$
Three	$x_1^3, x_2^3, x_1^2x_2, x_1x_2^2, x_1^2x_1^2, x_2^2x_1$

However, the PBW basis of A_2 up to PBW degree three is

PBW degree	Elements
Zero	1
One	$x_1, x_2, [x_1, x_2], [x_1, [x_1, x_2]], [x_1, [x_2, [x_1, x_2]]], \dots$
Two	$x_1^2, x_2^2, x_1x_2, x_1[x_1, x_2], [x_1, x_2][x_1, [x_2, x_1]], x_2[x_1, [x_2, [x_1, x_2]]], \dots$
Three	$x_1^3, x_2^3, x_1x_2x_1, [x_1, x_2][x_1, x_2]x_2, [x_1, [x_2, x_1], [x_1, x_2]]x_1[x_1, x_2], \dots$

The commutative ordering for the normal degree is not necessary, while it is for the PBW degree (due to the PBW theorem). The simplest case is $x_2x_1 = x_1x_2 - [x_1, x_2]$ where the error term $[x_1, x_2]$ is of lower PBW degree one.

Lemma 2.1 (The PBW argument). *For $r \geq 2$ and $k \geq 0$ any element m in $L_r(A_n)$ of standard degree $r + k$ has a PBW degree equal to $k + 1$.*

Proof. Due to bi-linearity it suffices to consider only a single commutator m with monomials in its slots (we can also ignore the scalars in \mathcal{K}). It is obvious that m is a pure commutator only for $k = 0$, because it has r slots and degree $r + k$. Hence when $k = 0$, $m \in \mathfrak{g}_n$ and thus m has a PBW degree one, which is $k + 1$ ($k = 0$).

When $k \geq 1$, w.l.o.g. in some of the slots of m there is a product of v generators $x_{i_1} \cdots x_{i_v}$, $v \geq 2$. Set $a := x_{i_1} \cdots x_{i_{v-1}}$ and $b := x_{i_v}$. Consider a product ab of elements a and b in a slot of m :

$$m = [a_1, [\dots, [a_{r'}, [ab, \ell]]] \dots] = \text{ad}(a_1)(\cdots \text{ad}(a_{r'}) (\text{ad}(ab)(\ell)) \cdots),$$

where ℓ is a commutator of length $r - r' - 1$. From the Leibniz rule $\text{ad}(ab)(\ell) = a\text{ad}(b)(\ell) + \text{ad}(a)(\ell)b$, and

$$m = \text{ad}(a_1)(\cdots \text{ad}(a_{r'}) (a\text{ad}(b)(\ell)) \cdots) + \text{ad}(a_1)(\cdots \text{ad}(a_{r'}) (\text{ad}(a)(\ell)b) \cdots)$$

Set $b' := \text{ad}(b)(\ell)$ and $c' := \text{ad}(a)(\ell)$. We reduce the number of adjoining operations by one and keep a product of two elements, namely ab' and $c'b$. By induction on r a product of two elements in a slot of m expands to a linear combination of commutators with products of less (normal) degree. This leads us to a product of $k + 1$ pure commutators in \mathfrak{g}_n , i.e. a product of $k + 1$ elements of PBW degree one. From here the statement of the lemma follows. \blacksquare

For instance, when $r = 3$ and $k = 1$, $[x_1, x_2, x_3x_4]$ expands to

$$[x_1, x_3][x_2, x_4] + x_3[x_1, x_2, x_4] + [x_2, x_3][x_1, x_4] + [x_1, x_2, x_3]x_4,$$

which has PBW degree two. We use Lemma 2.1 to prove non-containment of products of lower central series ideals. We know that $M_l(A_n) \cdot M_m(A_n) \subset M_{n+m-1-\epsilon}(A_n)$ where ϵ is zero or one, depending on the parity of l and m . It is natural to conjecture that $M_l \cdot M_m$ is not in M_{l+m} . The idea is to suppose this is true, and take the lowest degree part of the LHS and the RHS. The lowest degree part of a lower central series ideal is a pure commutator. Thus the LHS has a product of two such commutators

that has PBW degree two while the RHS has a single pure commutator of PBW degree one. This is a contradiction. We extend this idea to the following result.

Proposition 2.2. *Given n greater than or equal to two and a k -tuple $i = (i_1, \dots, i_k)$ in \mathbb{N}^k the following holds*

$$I(A_n, i) \leq 1 - k + \sum_{p=1}^k i_p.$$

Proof. It suffices to show that

$$L_{i_1} \cdots L_{i_k} \not\subseteq M_{i_1 + \dots + i_k - k + 2}.$$

The lowest degree part of the LHS has the form $\ell_1 \cdots \ell_k$, where $\ell_j \in \mathfrak{g}_n$ is a pure commutator in L_j for $j = 1, \dots, k$. Therefore, the LHS has a PBW degree k .

The lowest degree part of the RHS, w.l.o.g., has the form $a \cdot \ell$, where $\ell \in L_{i_1 + \dots + i_k - k + 2}$. The normal degree is $\sum_{p=1}^k i_p$, due to the LHS. Hence, we may think we start from a pure commutator and insert $k - 2$ generators in a or in the slots of the commutator. If the normal degree of a is p ($p = 0, 1, \dots, k - 2$) the normal degree of ℓ is $\sum_{j=1}^k i_j + k - 2 - p$. Now, we apply Lemma 2.1 to ℓ to prove that the RHS breaks into components of PBW degree less than or equal to $k - 1$. This is incompatible with the PBW degree of LHS. \blacksquare

Remark 2.3. Note that this is the first general result to give us an upper bound on the function I .

3. MAXIMAL CONTAINMENT FOR $n = 2$. GENERATORS OF M -IDEALS

Our proofs of containment rely on the following three identities true for all free associative unitary algebras:

- $[ab, c] = a[b, c] + [a, c]b$ — The Leibniz Law;
- $[a, b, c] + [c, a, b] + [b, c, a] = 0$ — The Jacobi Identity;
- $[a, b][a, c] = a[a, b, c] + [a, b, a]c + [a, ac, b]$ — The Pigeonhole Identity ¹.

3.1. Proof of Theorem 1.7. The PBW argument gives us the upper bound. It suffices to show that it is sharp, i.e. to prove the following result.

Theorem 3.1 (Containment for $n = 2$). *For i and j greater than or equal to two*

$$M_i(A_2) \cdot M_j(A_2) \subset M_{i+j-1}(A_2).$$

In this section we set $M_i \equiv M_i(A_2)$ for any i and $A \equiv A_2$. We start with a lemma that relies on the pigeonhole principle.

Lemma 3.2. $[M_k, L_1] \subset M_{k+1}$ for $k \geq 2$.

Proof. When k is odd the proof follows from Bapat and Jordan's result (in this case we even have the stronger condition $[M_k, L_1] \subset L_{k+1}$). Hence, it suffices to consider k even. Take a general generator $m = [a[b, l], c] \in [M_k, L_1]$ where $a, b, c \in A$ and $l \in L_{k-1}$. Then

$$m = [b, l][a, c] + a[[b, l], c] = [b, l][a, c] \pmod{M_{k+1}}.$$

If a is in the field \mathcal{K} then $[a, c] = 0$, i.e. $m = 0 \pmod{M_{k+1}}$. Otherwise, due to bi-linearity let a and c be monomials. Similarly, w.l.o.g. b is a monomial. If a

¹This identity can be derived from expansion of $[a, ab, c]$ via the Leibniz rule.

and c are monomials in only one generator then $m = 0 \pmod{M_{k+1}}$. Otherwise their expressions as products of generators differ by at least one generator. Hence, since $n = 2$ b shares a generator with either a or c . Due to anti-commutativity the common generator is between b and a and w.l.o.g. suppose it is x_1 .

Further suppose $b = b'x_1b''$, and $a = a'x_1a''$. Using the Leibniz rule we can reduce the degree of b and a by removing the *extra* parts b', b'', c', c'' . Induction on the degree of the commutators provides us with the following transformation

$$(1) \quad [b, l][a, c] \longrightarrow [x_1, l]d[x_1, c],$$

where due to the multilinearity, w.l.o.g. we set $d \in A_2$ to be a monomial. Using the Leibniz we obtain:

$$(2) \quad [x_1, l]d[x_1, c] \longrightarrow l[x_1, d][x_1, c] + [x_1, ld][x_1, c]$$

We apply the pigeonhole identity to the RHS of Equation (2) to get products of the following kinds (we ignore scalars, signs, and due to Leibniz we can multiply by L_1 only to the left side):

- $l(x_1[x_1, d, c] + [x_1, d, x_1]c + [x_1, x_1c, d]) \in L_{k-1} \cdot M_3 \subset M_{k+1}$;
- $x_1[x_1, ld, c] + [x_1, ld, x_1]c + [x_1, x_1c, ld] \in L_1 \cdot [L_1, M_{k-1}, L_1] \subset L_1 \cdot [L_1, L_k] \subset M_{k+1}$,

because $k - 1$ is odd and Bapat and Jordan's $[M_{k-1}, L_1] \subset L_k$ holds. ■

Note that, due to the pigeonhole principle, a simple case of this lemma, $[M_2, L_1] \subset M_3$ is true for all algebras that admit $M_2 \cdot M_2 \subset M_3$. We conjecture a stronger version of the lemma (although not particularly needed in our proof).

Conjecture 3.3. $[M_k, L_1] = L_{k+1}$

Theorem 3.4. For any k and j $[M_k, L_j] \subset M_{k+j}$.

Proof. We use induction on j . The base case $j = 1$ follows from Lemma 3.2. Suppose the $[M_k, L_{j'}] \subset M_{k+j'}$ for any $j' \leq j$ and k arbitrary. Consider a general generator $m = [m', [a, l]]$ of $[M_k, L_{j+1}]$, where $m' \in M_k$, $a \in L_1$, and $l \in L_j$. By the Jacobi identity and anti-commutativity $m = [[m', a], l] + [[l, m], a]$. From Lemma 3.2 $[m', a] = 0 \pmod{M_{k+1}}$, hence by assumption $[[m', a], l] = 0 \pmod{[M_{k+1}, L_j] = 0 \pmod{M_{k+j+1}}}$. Similarly $[l, m] = 0 \pmod{M_{k+j}}$, i.e. $[[l, m], a] = 0 \pmod{M_{k+j+1}}$. Hence, $m = 0 \pmod{M_{k+j+1}}$. ■

To complete the proof of Theorem 1.7 we have that $M_i \cdot M_j \subset M_{i+j-1}$ when i or j is odd (Bapat and Jordan). So assume that i and j are even. It suffices to show that $L_i \cdot L_j \subset M_{i+j-1}$. Take $a \in A$, $b \in L_{i-1}$ and $c \in L_j$. Then $m = [b, a]c$ is a general generator of $L_i \cdot L_j$. By Leibniz $m = [b, ac] - a[b, c]$. Theorem 3.4 applied to the RHS finishes the proof. ■

3.2. Generators of M -ideals. An application of maximal containment.

By Theorem 1.7 we get the equality

$$I(A_2, (i_1, \dots, i_k)) = i_1 + \dots + i_k - k + 1.$$

Note that this is the first result that explicitly obtains $I(A_2, i)$. Moreover, in this section we apply this result to find a complete classification of the set of generators

of the ideals $M_i(A_2)$ for $i \geq 2$. Please recall the definition of pure-commutator product from the preliminaries.

First we note two observations:

- **Pure-commutator Decomposition Property** – every commutator of any length can be presented as a linear combination of pure-commutator products of the PBW degree of the commutator. This follows by the Leibniz rule and induction. For example $[x_1, x_2x_3, x_4] = [x_1, x_2][x_3, x_4] + x_2[x_1, x_3, x_4] + [x_1, x_2, x_4]x_3 + [x_2, x_4][x_1, x_3]$.
- **Free Permutation Property** – the commutators in every pure-commutator product can be permuted at the cost of an error term of higher degree. This is due to the definition of the bracket and the well-known fact (follows by Jacobi identity and induction) that $[L_i, L_j] \subset L_{i+j}$. We call it “free permutation,” because the error term is zero module most of the ideals in the considerations that follow. For example $[x_1, x_2][x_3, x_4] = [x_3, x_4][x_1, x_2] \bmod M_3$, because $[x_3, x_4][x_1, x_2] = [x_1, x_2][x_3, x_4] - [[x_1, x_2], [x_3, x_4]]$.

Theorem 3.5 (Finite Generation). *For $i \geq 2$ the ideal $M_i(A_2)$ is finitely generated as a two-sided ideal by the set \mathcal{S}_i of all pure-commutator products, consisting of commutators of length greater than two, that have a maximal inclusion in $M_i(A_2)$.*

Proof. It suffices to show that

$$\langle \mathcal{S}_i \rangle := \{alb \mid a, b \in A_2, \ell \in \mathcal{S}_i\} = M_i(A_2).$$

Take an element $m = alb$ from $\langle \mathcal{S}_i \rangle$ $a, b \in A_2, \ell \in M_i(A_2)$. By definition $\ell \in M_i(A_2)$, hence by definition of $M_i(A_2)$ $m \in A_2M_i(A_2)A_2 = M_i(A_2)$. Therefore, $\langle \mathcal{S}_i \rangle \subset M_i(A_2)$.

To show the reversed inclusions take an element $m \in \mathcal{S}_i$. Through the Pure-commutator Decomposition we present this element as a linear combination of pure-commutator products. Without loss of generality, we consider one such product and use the Free Permutation to present it of the form

$$m = \alpha(x_{j_1}x_{j_2} \cdots x_{j_p})\ell_1\ell_2 \cdots \ell_q(x_{k_1}x_{k_2} \cdots x_{k_r}),$$

where $\alpha \in \mathcal{K}$, ℓ_s is a pure commutator of length i_s for $1 \leq s \leq q$, and p, q, r are some finite positive integers. We ignore the scalar α as $\mathcal{K} \subset A_2$. Note that the transformed m is a pure-commutator product of $p + q + r$ pure commutators, and its associated tuple is $(1^q, i_1, \dots, i_s, 1^r)$. Hence, due to the PBW Argument, m is maximally included in $M_{i_1 + \dots + i_q + q + r - (p + q + r) + 1} = M_{i_1 + \dots + i_q - q + 1}$.

Now we consider cases for the sum $s = i_1 + \dots + i_q - q + 1$:

- $s < i$: Then m is not in $M_i(A_2)$, which is a contradiction with its choice;
- $s = i$: It means that m is a pure-commutator product that has a maximal inclusion in $M_\Sigma = M_i$, i.e. $m \in \langle \mathcal{S}_i \rangle$;
- $s > i$: We argue that we can reduce it to the case $s = i$. Since an element from $\langle \mathcal{S}_i \rangle$ is multiplied on both sides by A_2 , w.l.o.g. we consider only $\ell_1\ell_2 \cdots \ell_q$. Due to the same reason, that we can multiply by A_2 elements to the sides, it suffices to consider the case

$$I(A_2, (i_1, \dots, i_q)) > i \text{ and } I(A_2, (i_2, \dots, i_q)) < i.$$

But, since the length of ℓ_1 is greater than one, we can present it as $\ell_1 = [a, \ell]$, where $a \in A_2$ and ℓ is another pure-commutator. Then by simple expansion

we get

$$\ell_1 \ell_2 \cdots \ell_q = a \ell_2 \cdots \ell_q - \ell a \ell_2 \cdots \ell_q,$$

but through Free Permutation we can “push” the element a in the second summand to the farthest left to obtain

$$a \ell_2 \cdots \ell_q - \ell \ell_2 \cdots \ell_q a.$$

This sum evaluates under the function I to $s-1$. We permuted commutators in exchange of an error term, i.e. we get the combination of the two commutators, which is a pure commutator of the combined degree. However, in this case we reduce q by one, so we can proceed with double induction on q and s . The base of the induction is $q = 1$ and $s = \ell_1$. Then we have the commutator $[a_1, \dots, a_s] = [a_1, \dots, a_{s-i}, [a_{s-i+1}, \dots, a_s]] \in A_2 \cdot \{[a_{s-i+1}, \dots, a_s]\} \cdot A_2 \subset \langle \mathcal{S}_i \rangle$, obviously. We repeat until we get to $s = i$. ■

By convention we set $\mathcal{S}_1 := \{1\}$. Obviously this generates $M_1 \equiv A$ as a two-sided ideal.

3.3. An application to Universal Lie Nilpotent Associative Algebras. Universal Lie Nilpotent Associative algebras are first introduced by Etingof, Kim and Ma [6], and are defined by $Q_i(A_n) := A_n/M_i(A_n)$. The algebra Q_i is the most free algebra that satisfies $[a_1, \dots, a_i] = 0$. The paper motivates the need to study the Q -algebras, because they help us better understand Lie nilpotent algebras B of any index i (i.e. $M_{i+1}(B) = 0$). In particular, Etingof, Kim and Ma give a description of $Q_4(A_n)$ in terms of generators and relations. We follow the ideas in the paper to obtain a description of $Q_i(A_2)$ for any $i \geq 2$.

Theorem 3.6. *For any $i \geq 2$ the ideal $M_i(A_2)$ is generated by the Lie relations $P = 0$ for any $P \in \mathcal{S}_i$. Moreover, $Q_i(A_2)$ and $A_2/\langle \mathcal{S}_i \rangle$ are isomorphic (as vector spaces).*

Proof. Let $C_i^{(2)} := A_2/\langle \mathcal{S}_i \rangle$. All of the relations $P = 0$ follow from Theorem 1.7. Hence, there is a natural surjective homomorphism $\theta : C_i^{(2)} \rightarrow Q_i(A_2)$. We are left to show that θ is an isomorphism. It suffices to prove that $[a_1, \dots, a_{i-1}]$ is a central element in $C_i^{(2)}$. We have that $[a_1, \dots, a_{i-1}] = 0$ in $Q_{i-1}(A_2)$. Therefore, $[a_1, \dots, a_{i-1}] \in M_{i-1}(A_2)$. Hence, we need to show that the generators of $M_{i-1}(A_2)$ are central. We know the generating set by Theorem 3.5. Thus, take $P' \in \mathcal{S}_{i-1}$. W.l.o.g. to prove centrality we consider only $[P', x_1]$. Due to the Pure-Commutator Decomposition Property we expand $[P', x_1]$ as a linear combination of pure-commutator products P , where P is obtained from P' by extending one of the commutators by one slot, i.e. P evaluates to i under I . Hence, $[P', x_1] = 0$ in $C_i^{(2)}$, which completes the proof. ■

4. DISCUSSION ABOUT MAXIMAL CONTAINMENT FOR $n = 3$.

This is a good place to recap the main results of the paper so far. On the one hand, we gave the PBW argument that presents us with an upper bound on $I(A_n, i)$. On the other hand, for the special case of $n = 2$ we used the Pigeonhole identity to set a lower bound on $I(A_2, i)$ and saw that this is sharp. This lead us to the maximal inclusions of lower central series ideals for $n = 2$.

It is tempting to find a sharp lower bound for $n = 3$ too. Indeed, the Pigeonhole identity gives us $M_2(A_3) \cdot M_2(A_3) \subset M_3(A_3)$. Moreover, by Theorem 1.3 we have that $M_i \cdot M_j \subset M_{i+j-1}$ for i or j odd. Hence, it suffices to consider i and j even. The first open question is whether $M_2(A_3) \cdot M_4(A_3)$ lives in $M_5(A_3)$.

- $[x, y][z, [x, [x, y]]] \in M_5$ (this is due to Jacobi identity applied on the second commutator and the Pigeonhole identity);
- $[x, y][z, y, x, y] \in M_5$: The proof is as follows. Expanding the second commutator via Jacobi identity we get (up to sign) $[[x, y], [z, y]]$ and $[y, [[x, y], z]]$. The latter commutator of the expansion allows us to use the Pigeonhole identity. The former is a bit trickier. Consider $[x, y][[x, y], [z, y]]$. Note that, due to the Leibniz rule, $a[a, b] = [a, ab]$ for any a and b . Then $[x, y][[x, y], [z, y]] = [[x, y], [x, y][z, y]] \in [L_2, M_3] \subset M_5$. We used the Pigeonhole identity and Bapat and Jordan's observation. ■
- One can easily give an argument (reduction of degree via the Leibniz rule) that to prove $M_i \cdot M_j \subset M_k$ for some i, j, k and any algebra admitting the Leibniz rule, it is sufficient to do it for pure commutators only.

The stumble is the following commutator $[x, y][z, z, z, x]$. Is it in M_5 ? We tried to use the trick from the bullet points but we end up with the same thing. The problem is that the common generator, x , is “farther down to the left.” The ultimate goal is to bring it as close as possible to the other x so that we can apply the Pigeonhole identity. We believe this is not a trivial task, and maybe more identities would be needed.

However, computer calculations for small degree tell us that $[x, y][z, z, z, x]$ is in M_5 . More generally we conjecture the following:

Conjecture 4.1. *Given the k -tuple $i = (i_1, \dots, i_k)$, such that i_p is greater than or equal to two for $p = 1, \dots, k$ the following holds*

$$I(A_3, i) = i_1 + \dots + i_k - k + 1.$$

The significant connection to the case $n = 2$ is that we have a small amount of generators, which allows us to have a common generator between two commutators.

5. AN APPROACH TO NON-CONTAINMENT

In order to improve the classification of lower central series containments, we need to find $I(A_n, (i, j))$ when i and j are both even. We know that in this case $M_i \cdot M_j \subset M_{i+j-2}$. Etingof, Kim, and Ma [6] conjectured that—generally— $M_i \cdot M_j \subset M_{i+j-1}$ if and only if i or j is odd (see Conjecture 3.6 in the arXiv paper). In 2010 Bapat and Jordan [1] prove the “if” part of the statement (see Corollary 1.4 in the arXiv paper). Hence, the conjecture boils down to showing that $I(A_n, (i, j)) = i + j - 2$ when i and j are both even. This is an interesting glitch of the general pattern. An approach might be to seek an associative algebra which does not admit the desired inclusion. If the algebra is chosen *properly*, by a standard mapping to the free associative algebra the conjecture would hold. In what follows we demonstrate our approach to the problem via the Lie algebra $\mathfrak{sl}(2)$ with its usual basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

Theorem 5.1. *The following holds:*

$$M_i(A_n) \cdot M_j(A_n)$$

is not contained in $M_{i+j}(A_n)$ when i and j have the same parity for $n \geq 2$. Furthermore, the statement holds when i and j have different parity, but $n \geq 3$.

Proof. Consider the homomorphism

$$\varphi : A_2 \longrightarrow \text{Mat}_2(\mathcal{K})$$

defined by $\varphi(x_1) = e + f$ and $\varphi(x_2) = h$. We extend φ to a homomorphism

$$\varphi : A_n \longrightarrow \text{Mat}_2(\mathcal{K}).$$

Since the lowest degree elements lie in the L elements it suffice to show that the product $L_i(A_n) \cdot L_j(A_n)$ is not contained in $L_{i+j}(A_n)$. Take

$$l = \text{ad}^{i-1}(x_2)(x_1) \cdot \text{ad}^{j-1}(x_2)(x_1) \in L_i \cdot L_j.$$

Moreover, $\varphi(l) = \text{ad}^{i-1}(h)(e + f) \cdot \text{ad}^{j-1}(h)(e + f)$. Using the relations of $\mathfrak{sl}(2)$ and induction we obtain the following identities:

$$\text{ad}^k(h)(e + f) = \begin{cases} 2^k(e - f) & \text{for odd } k; \\ 2^k(e + f) & \text{for even } k. \end{cases}$$

Hence when i and j are both odd l maps to $2^{i+j}(e - f)^2$, and thus $\text{trace}(\varphi(l)) = 2^{i+j+1}$. Furthermore, when i and j are both even, l maps to $2^{i+j}(e + f)^2$, which also has nonzero $\text{trace}(\varphi(l)) = -2^{i+j+1}$. However, since commuting implies zero trace, all elements of $L_{i+j}(A_n)$ are mapped to matrices with trace equal to zero. This argument proves the statement for i and j of the same parity.

Moreover, if i and j are of different parity and $n \geq 3$, we can take the same homomorphism φ , but we map the generators differently:

$$\varphi(x_1) = e, \varphi(x_2) = h, \varphi(x_3) = f.$$

Now, the element $\text{ad}^{i-1}(x_2)(x_1)\text{ad}^{j-1}(x_2)(x_3)$ maps to a matrix that does not have trace zero. ■

Remark 5.2. The PBW argument (Proposition 2.2) is a stronger result, but the method in Theorem 5.1 is demonstrative. Furthermore, Deryabina and Krasilnikov [2] recently proved the “only if” part of the conjecture by showing—more generally—that $M_i(\mathcal{K}'\langle X \rangle) \cdot M_j(\mathcal{K}'\langle X \rangle) \not\subseteq M_{i+j-1}(\mathcal{K}'\langle X \rangle)$ if both i and j are even, where $\mathcal{K}'\langle X \rangle$ is the free unital associative algebra over \mathcal{K}' freely generated by the (infinite and countable) set of generators $X = \{x_1, x_2, \dots\}$. Note that \mathcal{K}' can be *any* field. In the special case $\mathcal{K}' := \mathcal{K}$ (of characteristic zero), taking a finite set of generators, they show that to prove the conjecture it suffices to use the algebra

$$\mathcal{A} := E \otimes E_{i+j-4},$$

where E is the infinite-dimensional unital Grassmann algebra and E_{i+j-4} is the unital Grassmann sub-algebra generated by the first $i+j-4$ generators. In fact the authors prove that $M_{i+j-1}(\mathcal{A})$ is zero, while $M_i(\mathcal{A}) \cdot M_j(\mathcal{A})$ is not. This statement fails for small numbers n of variables. In particular it requires $n \geq i+j$.

We conclude with a conjecture for the general case of tuples, consisting only of two's, i.e. $(2, \dots, 2) \in \mathbb{N}^k$. We believe this is true due to combinatorial considerations. We checked the validity of the conjecture for small cases.

Conjecture 5.3. *Given the k -tuple $j := (2, \dots, 2) \in \mathbb{N}^k$ consisting only of two's*

$$I(A_n, j) = \begin{cases} \max \left\{ 2 \left\lceil \frac{k}{2} - \frac{n}{4} \right\rceil, 0 \right\} + 2 & n \geq 4 \text{ and } n \text{ even;} \\ \max \left\{ 2 \left\lceil \frac{k}{2} - \frac{n}{4} + \frac{1}{4} \right\rceil, 0 \right\} + 2 & n \geq 5 \text{ and } n \text{ odd;} \\ k + 1 & \text{otherwise.} \end{cases}$$

Note that setting $i_1 = \dots = i_k = 2$ in Theorem 1.7 we get that $I(A_2, (i_1, \dots, i_k)) = I(A_2, (2, \dots, 2)) = 2k - k + 1 = k + 1$, which is compatible with the conjecture.

6. GENERAL BOUNDS ON $I(A_n, i)$. SUMMARY

We try to obtain some easy containments for an arbitrary long product of M -ideals, i.e. we consider $I(A_n, (i_1, \dots, i_k))$ for $k \geq 2$. Suppose that exactly p of the elements i_1, \dots, i_k are odd ($p \leq k$). Hence, we have $k - p$ even indices. We have Theorem 1.2 and Theorem 1.3 in our arsenal. These theorems take two ideals and include them in a bigger one. Therefore, we have k steps until we obtain the complete inclusion. On each step we take two indices and produce a new one. If we take two even indices, then we use Theorem 1.2 and get a new even index. This way after a step we have one even index less. If we have two odd indices, we use Theorem 1.3 and obtain an odd index, i.e. after the step we have one odd index less. If we have an odd index and an even index, we get an even index, i.e. after the step we have one odd index less. Therefore, when we use the stronger inclusion we always reduce the number of the odd indices by one. We start with p odd indices. It follows that

$$I(A_n, (i_1, \dots, i_k)) \geq i_1 + \dots + i_k - 2(k - 1) + p = \sum_{v=1}^k i_v - 2k + p + 1.$$

Note that the procedure is invariant of the order of the combinations of indices, because of the above-mentioned combinatorial considerations. This fact also follows algebraically by the Free Permutation Property. Moreover, if $p = k$ then due to the PBW argument we get a maximal inclusion.

In conclusion, we collect the bounds (and equalities) on I that follow from previous results, combined with this paper:

- $I(A_n, (m, l)) \geq m + l - 2$;
- $I(A_n, (m, l)) \geq m + l - 1$ when m or l is odd;
- $I(A_n, (i_1, \dots, i_k)) \leq i_1 + \dots + i_k - k + 1$;
- $I(A_2, (i_1, \dots, i_k)) = i_1 + \dots + i_k - k + 1$;
- $I(A_n, (i_1, \dots, i_k)) \geq i_1 + \dots + i_k - 2k + p + 1$ if p indices are odd;
- $I(A_n, (i_1, \dots, i_k)) = i_1 + \dots + i_k - k + 1$ if all indices are odd;
- $I(A_3, (2, 2)) = 3$.

7. APPLICATIONS TO PI -ALGEBRAS

It turns out that results on lower central series containments are very useful in the study of the following specific quotients

$$\{B_i(A) = L_i(A)/L_{i+1}(A)\}_{i \geq 1} \text{ and } \{N_i(A) = M_i(A)/M_{i+1}(A)\}_{i \geq 1}.$$

Finding the structure of these vector spaces is a hard open problem that involves the representation theory of W_n , the polynomial vector fields on \mathbb{C}^n , and, in similar language, the representation theory of the general linear group on n generators

GL_n . There are a few results that shed light on the structures of the B -series and the N -series ([7], [4], [6], [3]).

To apply the results from this paper we replace A with certain PI-algebras

$$R_{i,j}(A_n) := A_n / (M_i(A_n) \cdot M_j(A_n)) \text{ (we may omit } A_n \text{ when } n \text{ is implied).}$$

We similarly define the lower central series for $R_{i,j}$, i.e. $L_1(R_{i,j}) = R_{i,j}$ and $L_i(R_{i,j}) := [R_{i,j}, L_{i-1}(R_{i,j})]$. Hence, we obtain the analogous M , B and N ideal series. The PI-Algebras $A/(M_i(A) \cdot M_j(A))$ are of particular interest in this setting because the structure of the commutator ideals M_i allows the action of W_n to descend to these quotients. In particular, we hope to understand the lower central series quotients in terms of the representation theory of W_n .

We consider $R_{i,j}(A_n)$ as $A_n + \mathcal{I}_n$, where $\mathcal{I}_n := M_i(A_n) \cdot M_j(A_n)$ is the ideal that generates the equivalence classes. Then easily, just by playing with inclusions, one can show that Theorems 1.7 and 1.3 reveal that generally $B_i \cong N_i$ for many PI-algebras.

Theorem 7.1 (Isomorphism Property). *The following holds*

$$B_i(R_{j,2}) \cong N_i(R_{j,2}) \text{ as graded vector spaces}$$

for all positive integers $i \geq t$, where $t = 2 \lceil \frac{j+1}{2} \rceil$.

Moreover, PI-algebras allow us to obtain explicit description of the vector spaces N , which give us an explicit approximation of the corresponding PI-algebra. We use the identity

$$[a, b, \ell] = [b, a, \ell] + [[a, b], \ell],$$

which combined with the Jacobi identity and the anti-commutative law allows us to permute slots of the commutator with some error terms ($[[a, b], \ell]$) as a trade-off.

For example, we have such a description of the free metabelian associative algebra $R_{2,2}(A_n)$ (comprehensively studied by Drensky in [5]). We use the metabelian identity $[a, b][c, d] = 0$ to get that $[a, b, \ell] = [b, a, \ell] \pmod{M_2 \cdot M_2}$. This allows us to permute freely the elements in the commutator, and thus find a span for $M_i(R_{2,2})$. Induction leads us to the following result.

Theorem 7.2 (The Structure Theorem). *For a fixed $r \geq 1$ a basis for the elements $N_r(R_{2,2})$ is*

$$\{x_1^{a_1} \cdots x_n^{a_n} [x_{i_1}, \dots, x_{i_r}]\},$$

where $a_j \geq 0$ for all j , $i_1 > i_2$ and $i_2 \leq \dots \leq i_r$ in the commutators of length r .

Note that this is an explicit description of the structure for the N -series of the free metabelian associative algebra, and it is for any number of generators n and any r . We extend this idea using maximal inclusions for $n = 2$ to prove other results, based on the language of GL_2 -representations.

Theorem 7.3 (The Structure Theorem for $R_{2,3}$). *For r greater than four we have that for the algebra $R_{2,3}(A_2)$ the following holds:*

$$N_r(R_{2,3}(A_2)) \cong \left(\sum_{i=0}^{\infty} Y_2(i) \right) \otimes (Y_2(r-1, 1) \oplus Y_2(r-3, 1) \otimes Y_2(1, 1)).$$

In this theorem $Y_2(r-1, 1)$ is the GL_2 module associated with the partition $(r-1, 1)$. We achieve this result through the maximal inclusion of the M -ideals and the following bijection map between “sorted” commutators, i.e. $i_1 > i_2 \leq i_3 \leq \dots \leq i_r$, and semi-standard Young tableaux:

$$[x_{i_1}, \dots, x_{i_r}] \mapsto \begin{array}{|c|c|c|c|} \hline i_2 & i_3 & i_4 & \cdots i_r \\ \hline i_1 & & & \\ \hline \end{array}.$$

Remark 7.4. Loosely speaking, in the above-mentioned problems we have two *parameters*: the number of generators, and the length of the commutators. It seems that if we fix one of the parameters to be sufficiently small we obtain results that provide us with a good classification of structure and behavior. We do not know how to think about explicit description when we set the two parameters free.

Finally, just for fun, we leave the discussion open with these two interesting and potentially strong identities.

$$\begin{aligned} [x, [y, [y, x]]] &= [y, [x, [y, x]]] \\ [xy, [y, x]] &= [yx, [y, x]] \end{aligned}$$

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